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SOLUTION OF THE STOCHASTIC BOUNDARY-VALUE PROBLEM OF STEADY-STATE CREEP FOR A THICK-WALLED TUBE USING THE SMALL-PARAMETER METHOD

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The physically and statistically nonlinear problem of steady-state creep for a thick-walled tube loaded by internal pressure is solved in the third approximation using the small-parameter method. The variances of random creep strain rates and displacements are calculated. The results obtained are compared with the solution of the same problem in the first and second approximations. A reliability assessment method for the tube using the strain failure criteria is proposed.

Key words: *stochastic heterogeneity, statistical nonlinearity, steady-state creep, thick-walled tube, boundary-value problem, small-parameter method.*

1. The substantial effect of random perturbations of the mechanical characteristics of materials on the stress and strain fields and the need for developing the corresponding stochastic models for strength analysis were discussed in many papers (see, e.g., [1–3]). This problem is of utmost importance for creep strain, for which the spread of experimental values is as high as 50–70% and one has to consider these results as acceptable [3–5].

Determining the strains and stresses of structural members subjected to nonlinear-creep conditions is a very difficult problem even in the deterministic formulation. The necessity of considering the microheterogeneities of the material leads to stochastic boundary-value problems, in which statistical nonlinearity should be taken into account in addition to the physical nonlinearity of the governing equations. Because of these difficulties, stochastic boundary-value creep problems admit analytical solutions only in the simplest cases [6–9].

To solve stochastic boundary-value problems in elastic and creep regions, the small-parameter method is used [6–10]. However, owing to substantial difficulties that arise in calculating the second and higher order moments of a random function, this method provides solutions of the stochastic boundary-value problems of steady-state creep only in the first approximation [6, 8].

In the present paper, the analytical solution of the boundary-value problem of steady-state creep of a thick-walled tube loaded by internal pressure is constructed to the third approximation using the small-parameter method.

We consider this problem in cylindrical coordinates for plane strain [$\varepsilon_z(r,t) = 0$ or $\dot\varepsilon_z(r,t) = 0$], assuming that the stochastic heterogeneities of the tube material are described by a function of one variable — the radius r. In this case, the components of the strain and stress tensors are random functions of the radius r.

In accordance to the theory of viscous flow (steady-state creep), the creep strains ε_r and ε_φ are described by the following rheological relations in stochastic form [9]:

$$\begin{gathered}\dot\varepsilon_r = -(\sqrt{3}/2)c(\sigma_\varphi - \sigma_r)^n[1 + \alpha U(r)],\\ \dot\varepsilon_\varphi = (\sqrt{3}/2)c(\sigma_\varphi - \sigma_r)^n[1 + \alpha U(r)].\end{gathered} \qquad (1)$$

Here σ_r and σ_φ are the radial and hoop stresses, respectively, $U(r)$ is the random function governing the stochastic heterogeneity of the tube material, whose statistical characteristics are known: $\langle U\rangle = 0$ and $\langle U^2\rangle = 1$, α is the

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coefficient of variation of the mechanical properties ($0 < \alpha < 1$), c and n are the material constants, and $\langle \cdot \rangle$ denotes the mathematical expectation.

The stresses σ_r and σ_φ satisfy the differential equation of equilibrium

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\varphi}{r} = 0 \tag{2}$$

and the boundary conditions

$$\sigma_r(a) = -q, \qquad \sigma_r(b) = 0, \tag{3}$$

where a and b are the inner and outer radii of the tube, respectively, and q is the pressure.

The strain-rate tensor components satisfy the compatibility condition

$$r\,\frac{d\dot{\varepsilon}_\varphi}{dr} + \dot{\varepsilon}_\varphi - \dot{\varepsilon}_r = 0. \tag{4}$$

We consider the problem of determining the stress–strain state of the tube, which reduces to solving system (1), (2), and (4) for the stresses subject to boundary conditions (3). This system can be reduced to a second-order statistically nonlinear equation for the radial stress (the prime denotes differentiation with respect to r):

$$r(1 + \alpha U(r))\sigma_r'' + \Big(\frac{n+2}{n}\,(1 + \alpha U(r)) + \frac{r}{n}\,\alpha U_r'\Big)\sigma_r' = 0. \tag{5}$$

To construct approximate analytical solutions of this equation, we expand the radial stress σ_r in power series of the small parameter α:

$$\sigma_r = \sigma_{r0} + \sum_{k=1}^{\infty} \alpha^k \sigma_{rk}, \qquad \langle \sigma_r \rangle = \sigma_{r0}. \tag{6}$$

Substituting (6) into Eq. (5) and equating the coefficients of the same powers of α, we obtain the system

$$r\sigma_{r0}'' + \frac{n+2}{n}\,\sigma_{r0}' = 0; \tag{7}$$

$$r\sigma_{r1}'' + \frac{n+2}{n}\,\sigma_{r1}' = -\frac{r}{n}\,U'\sigma_{r0}'; \tag{8}$$

$$\begin{gathered} r\sigma_{rk}'' + \frac{n+2}{n}\,\sigma_{rk}' = -\frac{r}{n}\,U'[\sigma_{r\,k-1}' - U\sigma_{r\,k-2}' + U^2\sigma_{r\,k-3}' - \ldots + (-1)^{k-1}U^{k-1}\sigma_{r0}'], \\ k = 2, 3, 4, \ldots . \end{gathered} \tag{9}$$

The solution of this system in recursive form involves computational difficulties. Therefore, we confine ourselves to the system of the first four equations. It comprises Eqs. (7) and (8) and the following two equations obtained from (9) for $k = 2, 3$:

$$r\sigma_{r2}'' + \frac{n+2}{n}\,\sigma_{r2}' = -\frac{r}{n}\,U'(\sigma_{r1}' - U\sigma_{r0}'); \tag{10}$$

$$r\sigma_{r3}'' + \frac{n+2}{n}\,\sigma_{r3}' = -\frac{r}{n}\,U'(\sigma_{r2}' - U\sigma_{r1}' + U^2\sigma_{r0}'). \tag{11}$$

System (7), (8), (10), (11) subject to boundary conditions (3) has the solution

$$\sigma_{r0} = A[b^{-2/n} - r^{-2/n}]; \tag{12}$$

$$\sigma_{r1} = \frac{2A}{n^2}\Big[(a^{-2/n} - r^{-2/n})H_1 - I_1(r)\Big]; \tag{13}$$

$$\sigma_{r2} = \frac{2A}{n^2}\Big[\frac{n+1}{2n}\,I_2(r) - \frac{2H_1}{n^2}\,I_1(r) + C_1(a^{-2/n} - r^{-2/n})\Big]; \tag{14}$$

$$\sigma_{r3} = \frac{2A}{n^2}\Big[-\frac{2n^2+3n+1}{6n^2}\,I_3(r) + \frac{(n+1)H_1}{n^3}\,I_2(r) - \frac{2}{n^2}\,C_1 I_1(r) + C_2(a^{-2/n} - r^{-2/n})\Big], \tag{15}$$

where

$$A = q/(a^{-2/n} - b^{-2/n}); \qquad H_k = BI_k(b) \quad (k = 1, 2, 3); \quad B = 1/(a^{-2/n} - b^{-2/n});$$

$$I_k(r) = \int_a^r U^k(x) x^{-1-2/n}\, dx \quad (k = 1, 2, 3);$$

$$C_1 = \frac{2H_1^2}{n^2} - \frac{n+1}{2n} H_2; \qquad C_2 = \frac{2n^2+3n+1}{6n^2} H_3 - \frac{n+1}{n^3} H_1 H_2 + \frac{2}{n^2} C_1 H_1.$$

Expression (12) is the well-known deterministic solution [11] and expressions (13)–(15) are solutions that correspond to the stochastic formulation of the problem. Thus, solution (12)–(15) determines the radial stress σ_r in the third approximation.

We now find approximate values of the strain-rate tensor components $\dot\varepsilon_r$ and $\dot\varepsilon_\varphi$ given by (1). Using solutions (12)–(15) and (2), the quantity $\sigma_\varphi - \sigma_r$ from relations (1) is written as

$$\sigma_\varphi - \sigma_r = r(\sigma'_{r0} + \alpha\sigma'_{r1} + \alpha^2\sigma'_{r2} + \alpha^3\sigma'_{r3}). \tag{16}$$

Raising the left and right sides of relation (16) to the nth power and substituting the resulting relation into (1), we obtain the component $\dot\varepsilon_\varphi$:

$$\dot\varepsilon_\varphi = r^n(\sigma'_{r0} + \alpha\sigma'_{r1} + \alpha^2\sigma'_{r2} + \alpha^3\sigma'_{r3})^n(1 + \alpha U).$$

Expanding the power function $(\sigma'_{r0} + \alpha\sigma'_{r1} + \alpha^2\sigma'_{r2} + \alpha^3\sigma'_{r3})^n$ in a Taylor series in α and retaining terms of up to the third order of smallness, after simple manipulations we obtain

$$\dot\varepsilon_\varphi = \frac{T}{r^2}\Big[1 + \frac{2\alpha}{n} H_1 + \frac{2\alpha^2(n+1)}{n^3} H_1^2 - \frac{\alpha^2(n+1)}{n^2} H_2 + \frac{\alpha^3(2n^2+3n+1)}{3n^3} H_3 - \frac{2\alpha^3(n+1)^2}{n^4} H_1 H_2 + \frac{4\alpha^3(n+1)(n+2)}{3n^5} H_1^3 + o(\alpha^3)\Big] = -\dot\varepsilon_r, \tag{17}$$

where $T = (\sqrt{3})^{n-1} c A^n / n^n$.

With allowance for (17), the displacement function becomes

$$u(t) = \varepsilon_\varphi r = (\dot\varepsilon_\varphi t) r = T\,\frac{t}{r}\Big[1 + \frac{2\alpha}{n} H_1 + \frac{2\alpha^2(n+1)}{n^3} H_1^2 - \frac{\alpha^2(n+1)}{n^2} H_2 + \frac{\alpha^3(2n^2+3n+1)}{3n^3} H_3 - \frac{2\alpha^3(n+1)^2}{n^4} H_1 H_2 + \frac{4\alpha^3(n+1)(n+2)}{3n^5} H_1^3 + o(\alpha^3)\Big]. \tag{18}$$

2. Let us find the statistical characteristics of the radial displacement $u(t)$. We calculate these characteristics assuming that the random function $U(r)$ governing the random field of perturbations of the mechanical properties of the material is distributed according to a normal law. In this case, the moments of odd orders vanish and the central moments of even orders are expressed in terms of the second-order moments. For example, the fourth-order moments are calculated by the formula [12]:

$$\langle \mathring{I}_1 \mathring{I}_2 \mathring{I}_3 \mathring{I}_4 \rangle = k_{12}k_{34} + k_{13}k_{24} + k_{14}k_{23}, \tag{19}$$

where $\mathring{I}_k$ are centered random quantities and k_{ij} are the second-order moments. All second-order moments are expressed in terms of the moments of the random function $I_k(r)$ as follows:

$$\begin{gathered}
\langle I_1(r)\rangle = \int_a^r \langle U(x)\rangle x^{-1-2/n}\, dx = 0,\\
\langle I_1^2(r)\rangle = \int_a^r\int_a^r \langle U(x_1)U(x_2)\rangle x_1^{-1-2/n} x_2^{-1-2/n}\, dx_1\, dx_2 = \int_a^r\int_a^r K(x_2 - x_1) x_1^{-1-2/n} x_2^{-1-2/n}\, dx_1\, dx_2,\\
\langle I_2(r)\rangle = \int_a^r \langle U^2(x)\rangle x^{-1-2/n}\, dx = \int_a^r x^{-1-2/n}\, dx = \frac{n}{2}(a^{-2/n} - r^{-2/n}),\\
\langle I_3(r)\rangle = \int_a^r \langle U^3(x)\rangle x^{-1-2/n}\, dx = 0,
\end{gathered} \tag{20}$$

where $K(x_2 - x_1)$ is a correlation function of the random homogeneous field $U(r)$.

Taking into account formulas (20), we write the mean displacements as

$$M_u = \langle u(t)\rangle = T\,\frac{t}{r}\Big[1 + \frac{2\alpha^2(n+1)\langle H_1^2\rangle}{n^3} - \frac{\alpha^2(n+1)\langle H_2\rangle}{n^2} + o(\alpha^3)\Big]. \tag{21}$$

Considering expressions (17) and (18) as the sums of dependent random functions, we obtain the variances of the random displacements and random strain rates

$$D_u = D[u(t)] = T^2\,\frac{t^2}{r^2}\Big[\frac{4\alpha^2}{n^2}\,D[H_1] + \frac{4\alpha^4(n+1)^2}{n^6}\,D[H_1^2] + \frac{\alpha^4(n+1)^2}{n^4}\,D[H_2]$$

$$+ \frac{\alpha^6(2n^2+3n+1)^2}{9n^6}\,D[H_3] + \frac{4\alpha^6(n+1)^4}{n^8}\,D[H_1H_2] + \frac{16\alpha^6(n+1)^2(n+2)^2}{9n^{10}}\,D[H_1^3]$$

$$+ \frac{4\alpha^4(2n^2+3n+1)}{3n^4}\,\langle \mathring{H}_1\mathring{H}_3\rangle - \frac{12\alpha^4(n+1)^2}{n^5}\,\langle \mathring{H}_1^2\mathring{H}_2\rangle + \frac{16\alpha^4(n+1)(n+2)}{3n^6}\,\langle \mathring{H}_1^4\rangle$$

$$- \frac{4\alpha^6(2n^2+3n+1)(n+1)^2}{3n^7}\,\langle \mathring{H}_1\mathring{H}_2\mathring{H}_3\rangle + \frac{8\alpha^6(2n^2+3n+1)(n+1)(n+2)}{9n^8}\,\langle \mathring{H}_1^3\mathring{H}_3\rangle$$

$$- \frac{16\alpha^6(n+1)^3(n+2)}{9n^9}\,\langle \mathring{H}_1^4\mathring{H}_2\rangle + o(\alpha^6)\Big]; \tag{22}$$

$$D[\dot\varepsilon_\varphi] = D[\dot\varepsilon_r] = D_u/(t^2r^2). \tag{23}$$

Using (19), we write each term in formulas (21)–(23) as follows:

$$D[H_1] = \langle H_1^2\rangle = B^2IK(n),$$

$$D[H_1^2] = \langle \mathring{H}_1^4\rangle = 3\langle H_1^2\rangle^2 = 3B^4(IK_1(n))^2, \qquad \langle H_2\rangle = B\langle I_2(b)\rangle = n/2,$$

$$D[H_2] = \langle \mathring{H}_2^2\rangle = B^2\int_a^b\int_a^b \langle U^2(x_1)U^2(x_2)\rangle x_1^{-1-2/n}x_2^{-1-2/n}\,dx_1\,dx_2 = \frac{n^2}{4} + 2B^2IK_2(n),$$

$$D[H_3] = \langle \mathring{H}_3^2\rangle = B^2\int_a^b\int_a^b \langle U^3(x_1)U^3(x_2)\rangle x_1^{-1-2/n}x_2^{-1-2/n}\,dx_1\,dx_2 = 9B^2IK_1(n),$$

$$D[H_1H_2] = \langle \mathring{H}_1^2\mathring{H}_2^2\rangle = B^4\int_a^b\int_a^b\int_a^b\int_a^b \langle U(x_1)U(x_2)U^2(x_3)U^2(x_4)\rangle$$

$$\times\, x_1^{-1-2/n}x_2^{-1-2/n}x_3^{-1-2/n}x_4^{-1-2/n}\,dx_1\,dx_2\,dx_3\,dx_4$$

$$= (n^2/4)B^2IK_1(n) + (n/2)B^3IK_3(n) + 2B^4(IK_1(n))^2 + 4B^4IK_4(n),$$

$$D[H_1^3] = \langle \mathring{H}_1^6\rangle = B^6\int_a^b\int_a^b\int_a^b\int_a^b\int_a^b\int_a^b \langle U(x_1)U(x_2)U(x_3)U(x_4)U(x_5)U(x_6)\rangle$$

$$\times\, x_1^{-1-2/n}x_2^{-1-2/n}x_3^{-1-2/n}x_4^{-1-2/n}x_5^{-1-2/n}x_6^{-1-2/n}\,dx_1\,dx_2\,dx_3\,dx_4\,dx_5\,dx_6 = 9B^6(IK_1(n))^3,$$

$$\langle \mathring{H}_1^2\mathring{H}_2\rangle = B^3\int_a^b\int_a^b\int_a^b \langle U(x_1)U(x_2)U^2(x_3)\rangle x_1^{-1-2/n}x_2^{-1-2/n}x_3^{-1-2/n}\,dx_1\,dx_2\,dx_3 = (n/2)B^2IK_1(n) + 2B^3IK_3(n),$$

$$\langle \mathring{H}_1 \mathring{H}_3 \rangle = B^2 \int_a^b \int_a^b \langle U(x_1) U^3(x_2) \rangle x_1^{-1-2/n} x_2^{-1-2/n} \, dx_1 \, dx_2 = 3B^2 IK_1(n),$$

$$\langle \mathring{H}_1^4 \rangle = B^4 \int_a^b \int_a^b \int_a^b \int_a^b \langle U(x_1) U(x_2) U(x_3) U(x_4) \rangle$$

$$\times x_1^{-1-2/n} x_2^{-1-2/n} x_3^{-1-2/n} x_4^{-1-2/n} \, dx_1 \, dx_2 \, dx_3 \, dx_4 = 3B^4 (IK_1(n))^2,$$

$$\langle \mathring{H}_1 \mathring{H}_2 \mathring{H}_3 \rangle = B^3 \int_a^b \int_a^b \int_a^b \langle U(x_1) U^2(x_2) U^3(x_3) \rangle$$

$$\times x_1^{-1-2/n} x_2^{-1-2/n} x_3^{-1-2/n} \, dx_1 \, dx_2 \, dx_3 = (3n/2) B^2 IK_1(n) + 6B^3 IK_3(n),$$

$$\langle \mathring{H}_1^3 \mathring{H}_3 \rangle = B^4 \int_a^b \int_a^b \int_a^b \int_a^b \langle U(x_1) U(x_2) U(x_3) U^3(x_4) \rangle$$

$$\times x_1^{-1-2/n} x_2^{-1-2/n} x_3^{-1-2/n} x_4^{-1-2/n} \, dx_1 \, dx_2 \, dx_3 \, dx_4 = 9B^4 (IK_1(n))^2,$$

$$\langle \mathring{H}_1^4 \mathring{H}_2 \rangle = B^5 \int_a^b \int_a^b \int_a^b \int_a^b \int_a^b \langle U(x_1) U(x_2) U(x_3) U(x_4) U^2(x_5) \rangle$$

$$\times x_1^{-1-2/n} x_2^{-1-2/n} x_3^{-1-2/n} x_4^{-1-2/n} x_5^{-1-2/n} \, dx_1 \, dx_2 \, dx_3 \, dx_4 \, dx_5$$

$$= (3n/2) B^4 (IK_1(n))^2 + 6B^5 IK_1(n) IK_3(n),$$

where

$$IK_1(n) = \int_a^b \int_a^b K(x_2 - x_1) x_1^{-1-2/n} x_2^{-1-2/n} \, dx_1 \, dx_2;$$

$$IK_2(n) = \int_a^b \int_a^b K^2(x_2 - x_1) x_1^{-1-2/n} x_2^{-1-2/n} \, dx_1 \, dx_2;$$

$$IK_3(n) = \int_a^b \int_a^b \int_a^b K(x_2 - x_1) K(x_3 - x_2) x_1^{-1-2/n} x_2^{-1-2/n} x_3^{-1-2/n} \, dx_1 \, dx_2 \, dx_3;$$

$$IK_4(n) = \int_a^b \int_a^b \int_a^b \int_a^b K(x_2 - x_1) K(x_3 - x_2) K(x_4 - x_3)$$

$$\times x_1^{-1-2/n} x_2^{-1-2/n} x_3^{-1-2/n} x_4^{-1-2/n} \, dx_1 \, dx_2 \, dx_3 \, dx_4.$$

3. As follows from the formulas given above, a relation for the correlation function should be given to calculate the variances.

Statistical processing of test data shows that the correlation functions of the mechanical characteristics are sign-variable decaying functions [13, 14] and can be approximated by the expression

$$K(\rho) = \mathrm{e}^{-\gamma|\rho|} (\cos(\beta\rho) + (\gamma/\beta) \sin \beta|\rho|), \qquad \rho = x_2 - x_1, \quad \gamma > 0, \tag{24}$$

where γ and β are constant quantities determined from the condition of the best fit to experimental data.

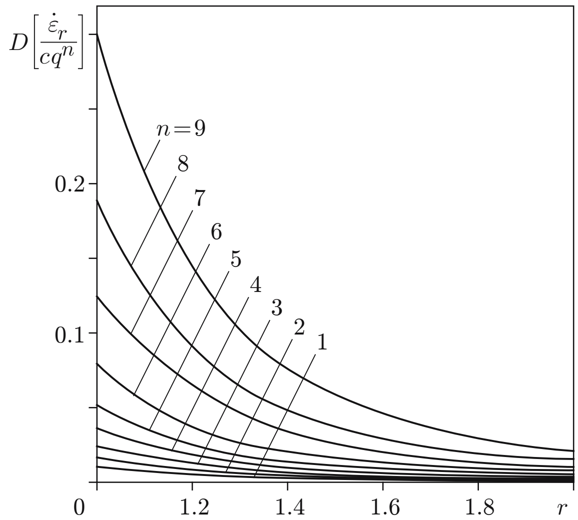

Fig. 1

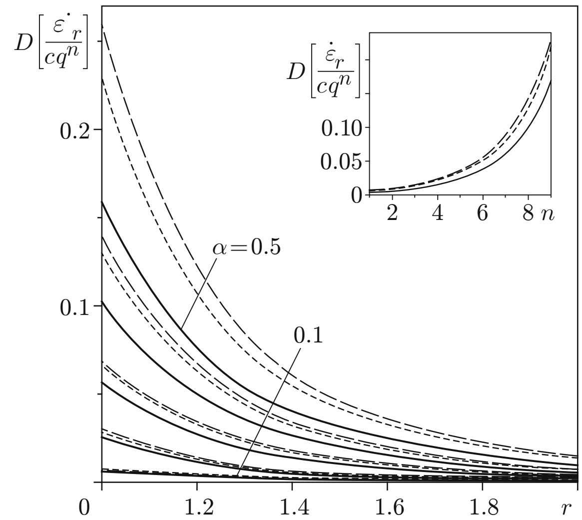

Fig. 2

Fig. 1. Variances of the reduced strain rates versus r for various n ($\alpha = 0.3$).

Fig. 2. Variances of the reduced strain rates in the first (solid curves), second (dotted curves), and third (dashed curves) approximations for various α ($n = 5$).

The second-order moments were calculated under the assumption that the correlation function of the random uniform and one-dimensional heterogeneity field $U(r)$ is given by (24) with the following numerical values of the parameters: $\gamma = 10$ and $\beta = 20$.

The goal of the further studies was to analyze the effect of the second and third approximations, the exponent n which takes into account the steady-creep nonlinearity, and the variation coefficient α on the variances of the strain rates $\dot{\varepsilon}_r$ and $\dot{\varepsilon}_\varphi$.

Numerical analysis of a thick-walled tube with inner and outer radii $a = 1$ and $b = 2$, respectively, shows that the variances of the reduced strain rates $D[\dot{\varepsilon}_r/(cq^n)]$ and $D[\dot{\varepsilon}_\varphi/(cq^n)]$ increase with n, the maximum variances occurring near the inner surface of the tube and the minimum values occurring near the outer surface. This finding is illustrated by plots of the variances as functions of the radius r (Fig. 1). Figure 2 shows the difference between the variances calculated in the first, second, and third approximations, which are represented by solid, dotted, and dashed curves, respectively. In the inset of Fig. 2, a section of the diagram shown in Fig. 1 is given for $r = 1.5$ and $\alpha = 0.3$, which shows the dependence of the variances on n, along with variance curves calculated in the second and third approximations.

Numerical values of the variances of the reduced strain rates are listed in Table 1 for various n and α and $r = 1.5$. The values calculated by retaining only the first term, the first two terms, and the first three terms in the expansion series are given in columns D1, D2, and D3, respectively.

One can see from Figs. 1 and 2 and Table 1 that, for slightly heterogeneous materials ($\alpha = 0.1$–0.2), the values of the strain-rate variances differ only slightly. For materials with a high degree of heterogeneity ($\alpha = 0.4$–0.5), the values of the strain-rate variances calculated in the third approximation can exceed those calculated in the second and first approximations by a factor of one and a half and two, respectively. In this case, the values of the strength and reliability margins of the thick-walled tube are highly overestimated if terms of the third order of smallness are ignored.

4. The performance of many structural members is estimated by parametric (strain) failure criteria. It is obvious that the reliability assessment of structural members using deterministic models is a first (and in some cases, unreliable) approximation and ignores the natural scatter of the mechanical characteristics and output parameters. The stochastic estimates of the creep strains and displacements obtained above allow one to solve the reliability problem of a thick-walled tube using the strain failure criterion in a statistical formulation.

TABLE 1

Variances of the Reduced Strain Rates
in the First (D1), Second (D2), and Third (D3) Approximations for Various n and α

α	n	D1	D2	D3
0.1	1	0.0006	0.0006	0.0006
	3	0.0013	0.0013	0.0013
	5	0.0030	0.0031	0.0031
	7	0.0074	0.0076	0.0076
	9	0.0179	0.0184	0.0185
	11	0.0437	0.0447	0.0451
0.2	1	0.0023	0.0028	0.0029
	3	0.0050	0.0057	0.0059
	5	0.0121	0.0135	0.0140
	7	0.0294	0.0325	0.0337
	9	0.0717	0.0788	0.0816
	11	0.1747	0.1914	0.1980
0.3	1	0.0051	0.0079	0.0085
	3	0.0113	0.0147	0.0159
	5	0.0272	0.0342	0.0367
	7	0.0663	0.0818	0.0877
	9	0.1613	0.1973	0.2114
	11	0.3930	0.4777	0.5115
0.4	1	0.0092	0.0178	0.0199
	3	0.0200	0.0309	0.0346
	5	0.0484	0.0704	0.0786
	7	0.1178	0.1668	0.1859
	9	0.2868	0.4004	0.4459
	11	0.6987	0.9664	1.0748
0.5	1	0.0141	0.0355	0.0408
	3	0.0313	0,0578	0,0671
	5	0.0757	0.1293	0.1497
	7	0.1841	0.3037	0.3513
	9	0.4482	0.7254	0.8383
	11	1.0918	1.7454	2.0154

We estimate the reliability of a thick-walled tube for the case where the service life is determined by the moment the displacement $u(t)$ reaches a certain value u_*.

Let the failure-free operation of the tube be given by the condition

$$u(t) < u_*,$$

where u_* is a specified deterministic quantity. In this case, the reliability function $P(t)$ governing the probability of failure-free operation in the interval $[0, t]$ is equal to the probability that the values of the random function $u(t)$ are in the admissible region $(0, u_*)$ within this time interval [1]:

$$P(t) = P\{u(\tau) \in (0, u_*),\ \tau \in [0, t]\}. \tag{25}$$

If the function $u(t)$ leaves the interval $(0, u_*)$ at a certain time, it cannot enter this interval again because the creep displacement is an increasing function. In view of this, we obtain the following simpler formula for the probability of failure-free operation $P(t)$ in the time interval $[0, t]$ [1]:

$$P(t) = P\{u(t) \in (0, u_*)\}. \tag{26}$$

Unlike in the general case (25), where it is necessary to consider the spikes of the random process in calculating the random function, in our case, it suffices to calculate the probability that the random function $u(t)$ is in the specified region at the given time using expressions (21) and (22) for the main characteristics of the displacement function $u(t)$.

To illustrate the reliability assessment method, we consider the creep of a pressurized thick-walled tube made of 12KhMF steel ($T = 590°\text{C}$) with material constants $c = 3.03 \cdot 10^{-14}$ and $n = 7.1$. The inner and outer radii

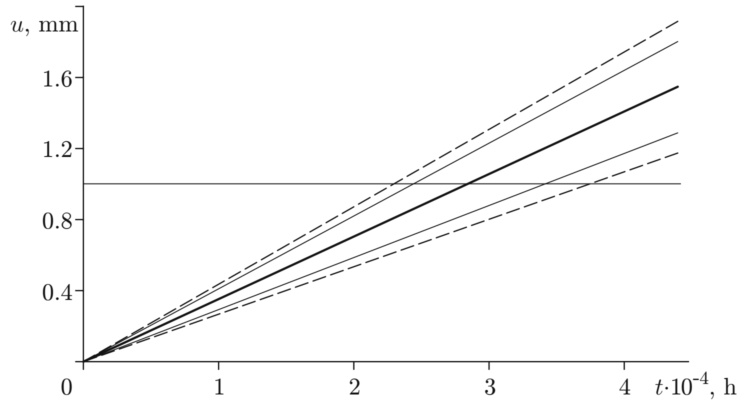

Fig. 3. Statistical estimate of the displacement of the inner surface of a thick-walled tube made of 12KhMF steel ($T = 590°$C) with inner and outer radii $a = 14$ mm and $b = 16.68$ mm, respectively, loaded by an internal pressure of $q = 28$ MPa.

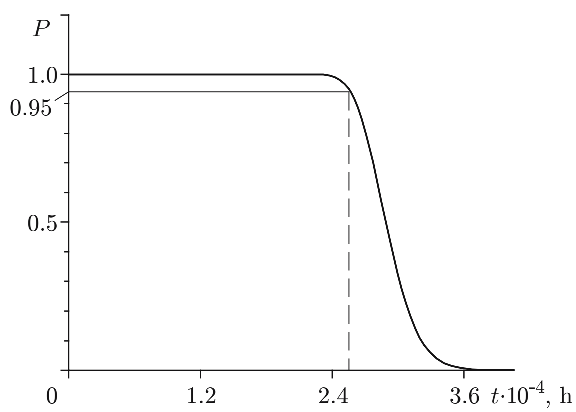

Fig. 4. Reliability function $P(t)$ for a thick-walled tube from 12KhMF steel ($T = 590°$C) with inner and outer radii $a = 14$ mm and $b = 16.68$ mm, respectively, loaded by an internal pressure of $q = 28$ MPa ($u_* = 1$ mm).

are $a = 14$ mm and $b = 16.68$ mm, respectively, the heterogeneity exponent is $\alpha = 0.3$, and the internal pressure is $q = 28$ MPa [15]. As a parameter determining the service life of the tube, we use the displacement of the inner surface, whose critical value is $u_* = 1$ mm.

The calculation yielded the following main characteristics of the random displacements of the inner surface: mathematical expectation $M_u = \langle u(t) \rangle = 3.52 \cdot 10^{-5}t$ and variance and root-mean-square deviation $D_u(t) = 3.78 \cdot 10^{-12}t^2$ and $s_u(t) = 1.946 \cdot 10^{-6}t$ for the first approximation and $D_u(t) = 7.82 \cdot 10^{-12}t^2$ and $s_u(t) = 2.796 \cdot 10^{-6}t$ $[s_u(t) = \sqrt{D_u(t)}\,]$ for the third approximation.

As an example, Fig. 3 shows the calculated values of the mathematical expectation for the displacement of the inner surface (solid thick curve) and intervals $u(t) \pm 3s_u(t)$ for the first approximation (solid thin curves) and third approximation (dashed curves).

The calculations show that for the given level of $u_* = 1$ mm, the mathematical expectation for the displacement $u(t)$ is reached after 28,431 h and its three-sigma band is 24,383–34,091 h for the first approximation and 22,956–37,337 h for the third approximation. It follows from the above example that the third approximation substantially refines the reliability estimate.

Using formula (26), the probability of failure-free operation is given by

$$P(t) = \frac{1}{\sqrt{2\pi}\, s_u(t)} \int\limits_0^{u_*} \mathrm{e}^{-(x-\langle u(t)\rangle)^2/(2s_u^2(t))}\, dx$$

or

$$P(t) = \Phi\Big[\frac{u_* - \langle u(t)\rangle}{s_u(t)}\Big] + \Phi\Big[\frac{\langle u(t)\rangle}{s_u(t)}\Big],$$

where $\Phi(x)$ is a Laplace function:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int\limits_0^{x} \mathrm{e}^{-z^2/2}\, dz.$$

The probability $P(t)$ can be used to determine the service life of a thick-walled tube. The design service life T_* is determined so that the probability of ensuring T_* is equal to the specified probability of failure-free operation p_*. In this case, the probability p_* should be reasonably close to unity.

Figure 4 shows the probability of failure-free operation of the tube versus time for $u_* = 1$ mm. One can see from Fig. 4 that for the given value $u_* = 1$ mm, the service life of the tube is $t = 25{,}143$ h for a probability of $p_* = 0.95$.

In summary, the proposed method for constructing approximate analytical solutions of stochastic boundary-value problems under nonlinear steady-creep conditions can be used to update existing models and solve the problem of assessing the reliability of cylindrical structural members.

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